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PROPAGATION OF ELASTIC WAVES ALONG A CYLINDRICAL CAVITY WITH A CONSTRAINED BOUNDARY†

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A generalization of the Biot problem [1] of the propagation of elastic waves along a cylindrical cavity with a free boundary is considered. It is assumed that there is a layer on the boundary described by boundary conditions of two kinds: (1) a Winkler-type foundation, and (2) a layer with inertial resistance. It is found that in the limiting case of a cavity of infinitely large radius or a frequency that increases without limit, the frequency equation is of the same accuracy as that for a half-space obtained in [2].

SUPPOSE the axis of a cylindrical cavity of radius a coincides with the z axis, the reaction of the medium to a radial perturbation is similar to the reaction of a Winkler foundation, and the shear stresses are zero. This can be either a continuous cylinder made of a Winkler-type material, filling the cylindrical cavity, or a layer which coats the cavity and has the properties of a linear reaction. Consider the propagation of a stationary elastic wave with a phase velocity p parallel to the z axis, which is the axis of symmetry of the motion. We will solve this problem using a scalar potential Φ and a vector potential Ψ, which satisfy the wave equations

$$\left(\nabla - \frac{1}{c_1^2}\partial_t^2\right)\Phi = 0, \quad \left(\nabla - \frac{1}{r^2} - \frac{1}{c_2^2}\partial_t^2\right)\Psi = 0$$
$$\nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{1}$$

where c_1 and c_2 are the velocities of propagation of longitudinal and transverse waves, respectively.

The displacements u_r and u_z and the non-zero stresses σ_{rr} and σ_{rz} are given by the relations

$$u_r = \frac{\partial\Phi}{\partial r} - \frac{\partial\Psi}{\partial z}, \quad u_z = \frac{\partial\Phi}{\partial z} + \frac{1}{r}\frac{\partial r\Psi}{\partial r}$$
$$\sigma_{rr} = 2\mu\left(\frac{\partial^2\Phi}{\partial r^2} - \frac{\partial^2\Psi}{\partial r\,\partial z}\right) + \frac{\rho(c_1^2 - 2c_2^2)}{c_1^2}\frac{\partial^2\Phi}{\partial t^2} \tag{2}$$
$$\sigma_{rz} = 2\mu\left(\frac{\partial^2\Phi}{\partial r\,\partial z} - \frac{\partial^2\Psi}{\partial z^2}\right) + \rho\frac{\partial^2\Psi}{\partial t^2}; \quad \mu = \rho c_2^2$$

We will seek the potentials Φ and Ψ in the form

$$\Phi = AH(r)\cos k(z - pt), \quad \Psi = BG(r)\sin k(z - pt)$$

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Substituting Φ and Ψ into the wave equations we obtain the expressions

$$\Phi = AK_0(\nu_1 r)\cos k(z-pt),\quad \Psi = BK_1(\nu_2 r)\sin k(z-pt)$$
$$\nu_i = k(1-\alpha_i^2)^{1/2},\quad \alpha_i = p^2/c_i^2,\ i=1,2 \tag{3}$$

where K_0 and K_1 are modified Bessel functions of the second kind, chosen taking the radiation condition at infinity into account.

The boundary conditions

$$r=a,\quad \sigma_{rr} = \eta u_r,\quad \sigma_{rz}=0$$

(η is the rigidity of the Winkler foundation), when relations (2) and (3) are taken into account, lead to a system of linear homogeneous equations for A and B. The condition for the solution of this system to be non-trivial gives the following characteristic equation

$$(2-\alpha_2^2)^2\xi_1 - 4\sqrt{1-\alpha_2^2}\sqrt{1-\alpha_1^2}\,\xi_2 - 2\alpha_2^2(ka)^{-1}\sqrt{1-\alpha_1^2} - \eta\alpha_2^2(k\mu)^{-1}\sqrt{1-\alpha_1^2}=0$$
$$\xi_i = K_0(\nu_i a)/K_1(\nu_i a),\ i=1,2 \tag{4}$$

which connects the phase velocity with the frequency $\omega = kp$ If the wavelength is very small, so that $\lambda/a\to 0$, then $ak = 2\pi a/\lambda\to\infty$, and Eq. (4), taking the asymptotic expression $K_n(z)\sim(\pi/(2z))^{1/2}e^{-z}$ into account reduces to the following form as $z\to\infty$

$$(2-\alpha_2^2)^2 - 4\sqrt{1-\alpha_1^2}\sqrt{1-\alpha_2^2} - \eta c_2\alpha_2^3(\omega\mu)^{-1}\sqrt{1-\alpha_1^2}=0 \tag{5}$$

Equation (4) differs from the frequency equation obtained by Biot in the presence of the last term, which changes the set of possible frequencies. Hence, for the limiting case, Eq. (5) differs from the Rayleigh equation to which the Biot equation reduces due to the presence of the last term. This limiting case [1] denotes that for very small wavelengths the curvature of the cavity can be ignored, i.e. the velocity of the wave propagation is determined as for an elastic half-space.

For a mixed-type support the boundary conditions have the form (η_1 is the rigidity of the foundation and m_1 is the inertial resistance)

$$r=a,\quad \sigma_{rr} = \eta_1 u_r + m_1\,\partial^2 u_r/\partial t^2,\quad \sigma_{rz}=0$$

As before, we obtain a frequency equation which differs from (4) in that η is replaced by $\eta_1 - m_1\omega^2$, and for the limiting case when $\lambda/a\to 0$, we obtain an equation which differs from (5) by the fact that $\eta c_2\alpha_2^3$ is replaced by $(\eta_1 - m_1\omega^2)p\alpha_2^2$.

The last equation does not allow [2] the existence of Rayleigh-type motion when

$$(\eta_1/\omega - m_1\omega)c_2 > \mu R(1)/(\alpha_2^3\sqrt{1-\alpha_1^2}) \equiv W$$

where R is the Rayleigh operator. This quadratic inequality for each fixed value of r has two roots, one of which is $\omega_k > 0$. For frequencies $\omega\le\omega_k$ Rayleigh motion does not occur (low frequencies are cut-off). Cut-off of low frequencies $(\omega\le\eta_1 c_2/W)$ also occurs when $m_1=0$. If $\eta_1=0$, there is no frequency cut-off.

The equation of the frequencies for a mixed type of support was solved numerically for $\eta_1=\eta_1^*=5.7\times10^9$ kg/m^3 and $m_1=m^*=1.4\times10^3$ kgf^2/m^3, which corresponds approximately to the modelling of a concrete tunnel of radius $a=3$ m and thickness $h=0.5$ m, situated in a granite medium ($\nu=0.3$, and $E=2.9\times10^{10}$ kg/m^2).

Figure 1 shows the phase velocity of the wave propagation p (divided by c_2) as a function of the wavelength λ (divided by the cavity diameter D). In all cases $p\to c_2$ as $\omega\to 0$ $(\lambda\to\infty)$. In the case of a Winkler foundation and when there is no inertial resistance, we cannot have $\alpha_2<0.928$, whereas for a rigidity η_1 equal to zero, all values of the phase velocity from 0 to c_2 are possible.

It can be seen that the relative velocity of the wave propagation α_2 in the case of a Winkler-type

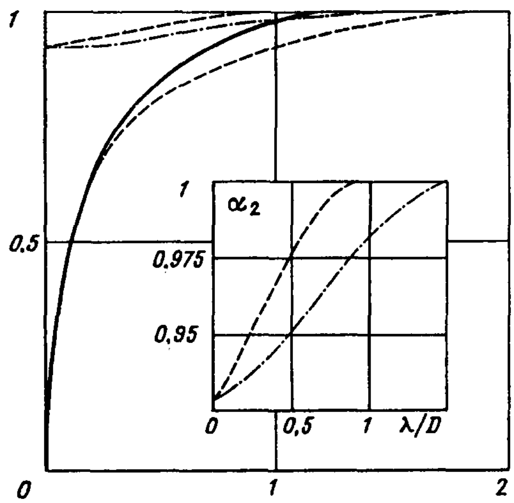

FIG. 1.

foundation when $\eta_1 = \eta_1^*$ and $m_1 = 0$ (the upper dashed curve) increases with λ / D much more rapidly than given by the Biot model with $\eta_1 = 0$, $m_1 = 0$ (the dash–dot curve).

For an inertial foundation (the lower dashed curve with $\eta_1 = 0$, $m_1 = m_1^*$) a qualitatively different relationship between α_2 and λ/D is found: there is no lower limit on the value of α_2.

The continuous curve ($\eta_1 = \eta_1^*$, $m_1 = m_1^*$) illustrates the combined effect of a Winkler and an inertial foundation on the value of α_2, and in this case there is no lower limit to the value of the velocity of the wave propagation p.

When there is inertial resistance ($m_1 = m_1^*$) a Winkler-type foundation does not change the curve of α_2 against λ/D; this was shown by comparing the two lower curves.

It should be noted that the presence of a Winkler support or inertial resistance gives a continuous frequency spectrum, unlike the previous case (a cavity of infinitely large radius)—the Rayleigh equation.

When

$$r = a, \quad \sigma_{rr} = 0, \quad \sigma_{rz} = \eta_2 u_z + m_2 \partial^2 u_z / \partial t^2$$

the frequency equation differs from (4) in that the last term on the left-hand side is replaced by

$$-(\eta_2 - m_2 \omega^2)(k\mu)^{-1} \alpha_2 \sqrt{1 - \alpha_2^2}\, \xi_1 \xi_2 + O(ak)$$

where

$$O(ak) = 2(\eta_2 - m_2 \omega^2)(\mu a k^2)^{-1} [\xi_2 \sqrt{1 - \alpha_1^2} \sqrt{1 - \alpha_2^2} - \xi_1] \to 0$$

as $ak \to \infty$.

In the limiting case of short waves, we have an equation which differs from (5) in having η replaced by $\eta_2 - m_2\omega^2$, which agrees with the equation investigated previously [2].

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